



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: May 1, 2015

REVISED: July 8, 2015

ACCEPTED: September 4, 2015

PUBLISHED: September 28, 2015

# Residual Weyl symmetry out of conformal geometry and its BRST structure

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**ABSTRACT:** The conformal structure of second order in  $m$ -dimensions together with the so-called (normal) conformal Cartan connection, is considered as a framework for gauge theories. The dressing field scheme presented in a previous work amounts to a decoupling of both the inversion and the Lorentz symmetries such that the residual gauge symmetry is the Weyl symmetry. On the one hand, it provides straightforwardly the *Riemannian parametrization* of the normal conformal Cartan connection and its curvature. On the other hand, it also provides the finite transformation laws under the Weyl rescaling of the various geometric objects involved. Subsequently, the dressing field method is shown to fit the BRST differential algebra treatment of infinitesimal gauge symmetry. The dressed ghost field encoding the residual Weyl symmetry is presented. The related so-called algebraic connection supplies relevant combinations found in the literature in the algebraic study of the Weyl anomaly.

**KEYWORDS:** Gauge Symmetry, Differential and Algebraic Geometry, Conformal and W Symmetry, BRST Symmetry

**ARXIV EPRINT:** [1504.08297](https://arxiv.org/abs/1504.08297)

*To the memory of Raymond Stora (1930–2015)*

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## 1 Introduction

The use of Cartan geometry in physics is ideally suited for the formulation of space-time transformations as gauge theories as well as for the study of geodesics, see for instance [1, 2]. General Relativity (GR) can be incorporated into the scheme of gauge theories under its Einstein-Cartan formulation [3, 4], see also [5] and for more insight on the subject see [6]. Relatively recent work [7–9] shows a renewed interest in the use of Cartan connections for gravity theories.

In particular, conformal gravity is based on the conformal group, which contains the Poincaré group, the Weyl symmetries (rescaling of a metric) and the special conformal transformations (inversions). Since the latter cannot be globally defined, gauging the inversions as local transformations is recommended. The mathematics of the second order conformal structure is well understood, see for instance [10–15], and, as a framework for gauge theories, it deserves to be studied for itself [16–18].

Regarding gauge symmetries in field theories, in [19, 20] was described a systematic procedure to reduce gauge symmetries. The advocated scheme turns out to be a redefinition of the field variables contained in the gauge theory at hand. It is grounded on the identification, among the fields of the theory, of what we called a *dressing field*.

As a continuation of [19] we provide in the current paper a new example of such a reduction of symmetries by the dressing field method applied to the second order conformal

structure as a framework for gravity gauge theories. Thanks to the dressing field method, we locally reduce the second order conformal structure to its Weyl subbundle i.e. reduction to the conformal Weyl symmetry.

Then, at the algebraic level, we show that the dressing field method is compatible with the usual Becchi-Rouet-Stora [21, 22] and Tyutin [23] (BRST symmetry) machinery developed for Yang-Mills theories, a cornerstone for quantizing gauge theories. This leads to the introduction of a *composite ghost* which handles the infinitesimal residual gauge freedom of the initial symmetry if any, and which enters into a “Russian formula” for the dressed connection.

We emphasize that the dressing field method has been systemized in [19] within the physicist’s language of gauge field theory based on differential forms. Accordingly, it is well-adapted to the algebraic differential structure of space of forms, connections and BRST algebras. On the contrary, it is not well-suited (yet) for dealing with differential operators such as covariant derivatives and the like. Conformal geometry, as an instance of parabolic Cartan geometry, is also studied as a tractor calculus which is based on the construction of a vector bundle associated with the Cartan geometry and a covariant derivative on its sections. Here we shall not be concerned with tractor calculus since our purpose is to treat, in a comparative way, one more explicit example of the dressing field method within the common language already used in [19, 24, 25]. However, as far as gauge field theory is concerned, both approaches are expected to be equivalent. Regarding the use of the tractor calculus, we refer to [26, 27] and references therein.

The paper is organized as follows. In section 2, the geometrical framework underlying the dressing field method is briefly recalled.

Then, in section 3, the dressing of a conformal Cartan connection is performed in two steps with suitable dressing fields. For the normal conformal Cartan connection, we end up with its so-called “Riemannian parametrization”, which straightforwardly provides known geometric objects relevant in conformal geometry [10]. Composition of dressing is shown to conspire in order to erase both the special conformal transformations (inversions) and the Lorentz  $SO(1, m - 1)$ -gauge symmetry. The remaining symmetry turns out to be given by the simple abelian group  $W \simeq \mathbb{R}_+ \setminus \{0\}$  of Weyl rescalings. This step by step reduction rests on the fact that the two needed dressing fields satisfy some compatibility conditions and can ultimately be merged together.

Section 4 investigates at the infinitesimal level the compatibility between the dressing field method and the elegant and efficient language of the BRST differential algebra [22, 28, 29]. A detailed account of the various modified BRST algebras coming out from the dressing scheme is given. Two examples illustrate this point. The first one concerns GR. The second one is the BRST treatment (linearized version) of the finite procedure presented in the first part of the paper concerning the second order conformal structure.

Section 5 gathers some concluding remarks.

## 2 The dressing field method in a nutshell

In this section, the main idea of the dressing scheme is introduced in a geometrical setting. To start with the basics, it is recalled that the usual geometrical framework for dealing with Yang-Mills theories is that of a principal bundle  $\mathcal{P} = \mathcal{P}(\mathcal{M}, H)$  over a space-time  $\mathcal{M}$ , with structure group  $H$ . Let us denote by  $\mathfrak{h}$  its Lie algebra. Let  $\omega \in \Omega^1(\mathcal{P}, \mathfrak{h})$  be a connection 1-form on  $\mathcal{P}$  and  $\Omega \in \Omega^2(\mathcal{P}, \mathfrak{h})$  its curvature; let  $\Psi$  denote a section of an associated bundle constructed out of a representation  $(V, \rho)$  of  $H$ .

Through a local trivializing section  $\sigma : U \subset \mathcal{M} \rightarrow \mathcal{P}$ , one gets the usual Yang-Mills gauge potential  $A := \sigma^*\omega$ , the field strength  $F := \sigma^*\Omega$  and the matter field  $\psi := \sigma^*\Psi$ . The local formulation on space-time will be understood with respect to the open set  $U$  throughout the paper.

To characterize the geometry of gauge fields, one has to specify the action of the (local) gauge group on each space of fields. The latter is defined by

$$\mathcal{H} = \{\gamma : U \subset \mathcal{M} \rightarrow H\},$$

with the group law inherited from  $H$ . The space  $\mathcal{H}$  can also be considered as a space of gauge fields for the following action of  $\gamma_2 \in \mathcal{H}$  (as a group) on  $\gamma_1 \in \mathcal{H}$  (as a space of fields):

$$\gamma_1^{\gamma_2} := \gamma_2^{-1} \gamma_1 \gamma_2, \quad (2.1)$$

which is compatible with the group law in  $H$ .

The field space  $\mathcal{A}$  of Yang-Mills potentials carries the usual action of the gauge group  $\mathcal{H}$ :  $A \mapsto A^\gamma := \gamma^{-1} A \gamma + \gamma^{-1} d\gamma$ , and accordingly,  $F \mapsto F^\gamma := \gamma^{-1} F \gamma$ . One has  $\psi \mapsto \psi^\gamma := \rho(\gamma^{-1})\psi$  for matter fields.

In the following, we consider new fields composed of “elementary” fields. For instance, taking  $A \in \mathcal{A}$  and  $\gamma_1 \in \mathcal{H}$  (as a space of fields), one constructs the “composed field”  $A^{\gamma_1} := \gamma_1^{-1} A \gamma_1 + \gamma_1^{-1} d\gamma_1$ . An induced action of the gauge group  $\mathcal{H}$  is naturally defined by taking the gauge group action on each of the elementary fields. As an illustration, for  $A^{\gamma_1}$  and for  $\gamma_2 \in \mathcal{H}$  (as a group) one has

$$\begin{aligned} (A^{\gamma_1})^{\gamma_2} &:= (A^{\gamma_2})^{\gamma_1^{\gamma_2}} = (A^{\gamma_2})^{\gamma_2^{-1} \gamma_1 \gamma_2} \\ &= (\gamma_2^{-1} \gamma_1 \gamma_2)^{-1} (A^{\gamma_2}) (\gamma_2^{-1} \gamma_1 \gamma_2) + (\gamma_2^{-1} \gamma_1 \gamma_2)^{-1} d(\gamma_2^{-1} \gamma_1 \gamma_2) \\ &= (\gamma_1 \gamma_2)^{-1} A (\gamma_1 \gamma_2) + (\gamma_1 \gamma_2)^{-1} d(\gamma_1 \gamma_2) = A^{(\gamma_1 \gamma_2)}, \end{aligned} \quad (2.2)$$

where  $A^{(\gamma_1 \gamma_2)}$  identifies with the action of  $\gamma_1 \gamma_2 \in \mathcal{H}$  (as a group) on  $A \in \mathcal{A}$ . Notice that this case is quite degenerate since  $A^{\gamma_1} \in \mathcal{A}$  and accordingly  $(A^{\gamma_1})^{\gamma_2} = \gamma_2^{-1} (A^{\gamma_1}) \gamma_2 + \gamma_2^{-1} d\gamma_2 = A^{(\gamma_1 \gamma_2)}$ . More subtle situations will be encountered in the following.

The dressing field method involves identifying, in a local trivialization of  $\mathcal{P}$ , a local dressing field in  $\mathcal{D} = \{u : U \rightarrow G'\}$  where  $G'$  is a target Lie group. This target group  $G'$  has to be chosen compatible with  $H$  and  $\mathcal{H}$  and their representations. In particular, one requires that there exists a subgroup  $H' \subseteq H$  such that  $\mathcal{D}$  supports the following action of the subgroup  $\mathcal{H}' := \{\gamma' : U \rightarrow H'\} \subseteq \mathcal{H}$ :

$$u \mapsto u^{\gamma'} := \gamma'^{-1} u.$$

Contrary to (2.1) this action is not compatible with the natural group law in  $G'$ . The clear distinction between  $\mathcal{H}$  and  $\mathcal{D}$  as  $\mathcal{H}'$ -gauge field spaces, is crucial, and should be kept in mind.

Another requirement on  $G'$  is the possibility to define, starting from the gauge fields  $A, F, \psi$ , and  $u$ , the following *composite fields*:

$$\hat{A} := A^u = u^{-1}Au + u^{-1}du, \quad \hat{F} := F^u = u^{-1}Fu, \quad \text{and} \quad \hat{\psi} := \psi^u = \rho(u^{-1})\psi. \quad (2.3)$$

It can be checked that  $\hat{F} = d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}]$ . These composite fields are readily seen to be  $\mathcal{H}'$ -gauge invariant ([19], Main Lemma).

At this stage, two possibilities are in order. First, if  $H' = H$  the above composite fields are  $\mathcal{H}$ -gauge invariant; the whole symmetry  $\mathcal{H}$  has thus been erased. Second, if  $H' \subsetneq H$ , then the composite field might display a residual gauge freedom: only the symmetry subgroup  $\mathcal{H}'$  has been neutralized. In either case, one can check indeed that, for instance (similarly to (2.2))  $(A^u)^{\gamma'} := (A^{\gamma'})^{u^{\gamma'}} = (A^{\gamma'})^{\gamma'^{-1}u} = A^u$ , for any  $\gamma' \in \mathcal{H}' \subseteq \mathcal{H}$ .

This construction turns out to be a geometrical root of the notion of “Dirac variables” [30, 31] and its generalization to non-abelian gauge field theories. As shown in [19] the method applies to the electroweak sector of the Standard Model (see also [24]) and to the Einstein-Cartan formulation of GR. In both cases it provided an interpretive shift with respect to the usual viewpoint. In [25] we showed how the method ought to be at the root of the so-called “Chen & al. trick” which has been sparking much work and reviving controversies on the nucleon spin decomposition issue.

### 3 Composing dressing fields

#### 3.1 The second order conformal structure

We refer to [15] for a detailed description of the Möbius geometry and its associated Cartan geometry. See also [10, 11] for a formulation in terms of higher order frame bundles. Notice that it is of prior importance to have a link between the second order conformal structure and a matrix representation. In the latter, one gains a direct contact with the usual formulation of a Yang-Mills gauge theory. Also, the matrix formulation allows easier calculations. We just provide the basic material necessary to our construction.

Let  $\mathcal{M}$  be a  $m$ -dimensional smooth manifold ( $m \geq 3$ ). Let  $CO(\mathcal{M})$  be the principal bundle of orthonormal frames with respect to the Minkowski  $\eta$  metric of signature  $(1, m-1)$ ,<sup>1</sup> with structure group

$$K_0 \simeq CO(1, m-1) = \{M \in GL_m(\mathbb{R}); M^T \eta M = z^2 \eta, z \in \mathbb{W}\} = SO(1, m-1) \times \mathbb{W}$$

where  $\mathbb{W} = \mathbb{R}_+ \setminus \{0\}$  is the group of Weyl rescaling. The Lie algebra of  $K_0$  is

$$\mathfrak{k}_0 := \mathfrak{co}(1, m-1) = \{v \in \mathfrak{gl}_m(\mathbb{R}); v^T \eta + \eta v = \epsilon \mathbf{1}_m, \epsilon \in \mathbb{R}\} = \mathfrak{o}(1, m-1) \oplus \mathbb{R}.$$

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<sup>1</sup>One could have worked with an arbitrary signature  $(p, q)$ , see e.g. [2].

The principal bundle  $CO(\mathcal{M})$  is a reduction of  $GL(\mathcal{M})$  the principal bundle of linear frames over  $\mathcal{M}$ . This is a first order  $G$ -structure which can be prolonged to a second order  $G$ -structure [11]. The latter is easier recast in the the following setting.

The Klein model geometry is the pair of Lie groups  $(G, H)$  where  $G = O(2, m)/\{\pm I_{m+2}\}$  with

$$O(2, m) = \left\{ M \in GL_{m+2}(\mathbb{R}) \mid M^T \Sigma M = \Sigma, \text{ where } \Sigma = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\}.$$

It is the isometry group of the de Sitter space  $dS^m$  defined as the quadric  $\Sigma(x, x) = 0$  in  $\mathbb{R}^{m+2}$ .  $H$  is the isotropy group of the point  $(1, 0, \dots, 0)$  such that  $dS^m \simeq G/H$  and has the following factorized matrix representation

$$H = K_0 K_1 = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & r & \frac{1}{2} r r^t \\ 0 & \mathbb{1} & r^t \\ 0 & 0 & 1 \end{pmatrix} \mid z \in \mathbb{W}, S \in SO(1, m-1), r \in \mathbb{R}^{m*} \right\},$$

where  $^t$  stands for the  $\eta$ -transposition, namely for the row vector  $r$  one has  $r^t = (r\eta^{-1})^T$  (where  $^T$  is the usual matrix transposition) and  $\mathbb{R}^{m*}$  is the dual of  $\mathbb{R}^m$ . Notice that  $K_1$  is an abelian subgroup of  $H$ . It can be shown that  $H \simeq CO(1, m-1) \ltimes \mathbb{R}^{m*}$ .

The infinitesimal Klein pair is  $(\mathfrak{g}, \mathfrak{h})$ , where both are graded Lie algebras [11]. They can respectively be decomposed according to  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathbb{R}^m \oplus \mathfrak{co}(1, m-1) \oplus \mathbb{R}^{m*}$ , a splitting which gives the different symmetry sectors: translations + (Lorentz  $\times$  Weyl) + inversions, and  $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathfrak{co}(1, m-1) \oplus \mathbb{R}^{m*}$ . The quotient space is just  $\mathfrak{g}/\mathfrak{h} =: \mathfrak{g}_{-1} \simeq \mathbb{R}^m$ . In matrix notation we have,

$$\begin{aligned} \mathfrak{g} &= \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ \tau & v & \iota^t \\ 0 & \tau^t & -\epsilon \end{pmatrix} \mid (v - \epsilon \mathbb{1}) \in \mathfrak{co}(1, m-1), \tau \in \mathbb{R}^m, \iota \in \mathbb{R}^{m*} \right\} \\ &\supset \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & v & \iota^t \\ 0 & 0 & -\epsilon \end{pmatrix} \mid \dots \right\}, \end{aligned}$$

with the  $\eta$ -transposition  $\tau^t = (\eta\tau)^T$  of the column vector  $\tau$ . The graded structure of the Lie algebras,  $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ ,  $i, j = 0, \pm 1$  with the abelian Lie subalgebras  $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0 = [\mathfrak{g}_1, \mathfrak{g}_1]$ , is automatically handled by the matrix commutator.

The second order conformal structure modelled on this Klein pair is a principal bundle,  $\mathcal{P}(\mathcal{M}, H)$ , with structure group  $H$ , together with a (local) Cartan connection  $\varpi \in \Omega^1(\mathcal{U}, \mathfrak{g})$  with curvature  $\Omega = d\varpi + \varpi^2 \in \Omega^2(\mathcal{U}, \mathfrak{g})$ . Accordingly, in matrix representation, the Cartan connection is parametrized by the matrix of 1-forms

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix},$$

where  $\theta$  is the soldering (or vielbein) 1-form which gives an isomorphism between each tangent space  $T_x\mathcal{M}$  and  $\mathfrak{g}_{-1} \simeq \mathbb{R}^m$ ; while the curvature is given by

$$\Omega = \begin{pmatrix} f & \Pi & 0 \\ \Theta & F & \Pi^t \\ 0 & \Theta^t & -f \end{pmatrix} := \begin{pmatrix} da + \alpha\theta & d\alpha + \alpha(A - a\mathbb{1}) & 0 \\ d\theta + (A - a\mathbb{1})\theta & dA + A^2 + \theta\alpha + \alpha^t\theta^t & d\alpha^t + (A + a\mathbb{1})\alpha^t \\ 0 & d\theta^t + \theta^t(A + a\mathbb{1}) & -da + \theta^t\alpha^t \end{pmatrix},$$

where the wedge product of forms is tacitly assumed.

The *normal conformal Cartan connection* is the unique  $\varpi$  (up to  $\mathcal{H}$ -gauge transformations) whose curvature is constrained by the two conditions [10, 11, 15]

$$i) \Theta \equiv 0, \text{ (torsion-free geometry), } ii) \text{Ric}(F) = F^a_{bac} \equiv 0, \text{ (Ricci-null condition), } \quad (3.1)$$

where the Einstein summation convention is understood and will be used throughout the paper. Combination of the latter with  $\mathfrak{g}_{-1}$ -sector of the Bianchi identity  $d\Omega + [\varpi, \Omega] = 0$  yields the traceless condition  $f = 0$ .

An element  $\gamma$  of the gauge group  $\mathcal{H}$  can be factorized as

$$\gamma = \gamma_0 \gamma_1 : \mathbb{U} \rightarrow H = K_0 K_1, \quad \text{with} \quad \begin{cases} \gamma_0 \in \mathcal{K}_0 := \{\gamma : \mathbb{U} \rightarrow K_0\}, \\ \gamma_1 \in \mathcal{K}_1 := \{\gamma : \mathbb{U} \rightarrow K_1\}. \end{cases}$$

Accordingly, with respect to  $\mathcal{K}_0$ , the gauge transformations of the Cartan connection are,

$$\begin{aligned} \varpi^{\gamma_0} &= \gamma_0^{-1} \varpi \gamma_0 + \gamma_0^{-1} d\gamma_0 \\ &=: \begin{pmatrix} a^{\gamma_0} & \alpha^{\gamma_0} & 0 \\ \theta^{\gamma_0} & A^{\gamma_0} & (\alpha^{\gamma_0})^t \\ 0 & (\theta^{\gamma_0})^t & -a^{\gamma_0} \end{pmatrix} = \begin{pmatrix} a + z^{-1}dz & z^{-1}\alpha S & 0 \\ S^{-1}\theta z & S^{-1}AS + S^{-1}dS & S^{-1}\alpha^t z^{-1} \\ 0 & z\theta^t S & -a + zdz^{-1} \end{pmatrix}, \end{aligned} \quad (3.2)$$

and with respect to  $\mathcal{K}_1$ ,

$$\begin{aligned} \varpi^{\gamma_1} &= \gamma_1^{-1} \varpi \gamma_1 + \gamma_1^{-1} d\gamma_1 =: \begin{pmatrix} a^{\gamma_1} & \alpha^{\gamma_1} & 0 \\ \theta^{\gamma_1} & A^{\gamma_1} & (\alpha^{\gamma_1})^t \\ 0 & (\theta^{\gamma_1})^t & -a^{\gamma_1} \end{pmatrix} \\ &= \begin{pmatrix} a - r\theta & ar - r\theta r + \alpha - rA + \frac{1}{2}rr^t\theta^t + dr & 0 \\ \theta & \theta r + A - r^t\theta^t & \theta \frac{1}{2}rr^t + Ar^t - r^t\theta^t r^t + \alpha^t + r^t a + dr^t \\ 0 & \theta^t & \theta^t r^t - a \end{pmatrix}. \end{aligned} \quad (3.3)$$

The principal bundle  $\mathcal{P}(\mathcal{M}, H)$  is a second order  $G$ -structure, a reduction of the second order frame bundle  $L^2\mathcal{M}$ ; it is thus a “2-stage bundle”. The bundle  $\mathcal{P}(\mathcal{M}, H)$  over  $\mathcal{M}$  can also be seen as a principal bundle  $\mathcal{P}_1 := \mathcal{P}(\mathcal{P}_0, K_1)$  with structure group  $K_1$  over  $\mathcal{P}_0 := \mathcal{P}(\mathcal{M}, K_0)$ , see [11]. The whole structure group  $H = K_0 K_1 \simeq CO(1, m-1) \ltimes \mathbb{R}^{m*}$  is of dimension  $1 + \frac{m(m-1)}{2} + m$ .

As shown in the next section, the symmetry group  $H$  can be reduced to the 1-dimensional Weyl group  $W$  through the dressing field method.

### 3.2 The need for two dressing fields

The second order bundle geometry  $\mathcal{P}_1 \rightarrow \mathcal{P}_0 \rightarrow \mathcal{M}$  suggests that the structure group  $K_1$  should be neutralized first, in order to reach the principal bundle  $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$ .

**First reduction.** In order to neutralize  $K_1$  (inversions) and so to ‘reduce’  $H = K_0 K_1$  to the  $K_0$  factor, we seek a dressing field, that is a local map, see [20],

$$u_1 : \mathcal{U} \rightarrow K_1 \quad \text{such that} \quad u_1^{\gamma_1} = \gamma_1^{-1} u_1, \text{ for any } \gamma_1 \in K_1,$$

whose matrix expression is

$$u_1 = \begin{pmatrix} 1 & q & \frac{1}{2} q q^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix},$$

where  $q : \mathcal{U} \rightarrow \mathbb{R}^{m*}$  is a covector field.

Such a dressing field can be extracted from a ‘gauge-fixing-like’ constraint,  $\chi(\varpi^{u_1}) = 0$ , imposed on the dressed Cartan connection  $\varpi_1$ . The vanishing of the  $(1, 1)$ -component (thus  $(3, 3)$  as well) of  $\varpi^{u_1}$  is taken to be such a constraint. On account of (3.3), it explicitly reads

$$\chi(\varpi^{u_1}) : a^{u_1} = a - q\theta = 0.$$

Solving the constraint for  $q$  leads to

$$a - q\theta = a - q_a \theta^a = a_\mu dx^\mu - q_a e^a_\mu dx^\mu = 0 \Rightarrow q_a = a_\mu (e^{-1})^\mu_a,$$

or in index free notation  $q = a \cdot e^{-1}$ , where “ $\cdot$ ” means greek index summation and will be used throughout the paper. In the latter, some care should be taken, bearing in mind that  $a$  is a covector, the *scalar coefficients* of the 1-form  $a$ . The distinction should be clear according to the context. Remark that the constraint yields a solution  $q$  which is *local* in the field theory sense, (i.e. a differential polynomial in the fields), contrary to the non local realisation of a dressing field as can be found in [30, 31].

Now, the  $\mathcal{K}_1$ -gauge transformation of the Cartan connection (3.3) gives us,

$$\begin{aligned} a^{\gamma_1} &= a - r\theta \rightarrow (a^{\gamma_1})_\mu = a_\mu - r_a e^a_\mu, & \text{or in index free notation} \quad a^{\gamma_1} &= a - re, \\ \theta^{\gamma_1} &= \theta \rightarrow (e^{\gamma_1})^a_\mu = e^a_\mu & \text{in index free notation} \quad e^{\gamma_1} &= e. \end{aligned}$$

This implies  $q^{\gamma_1} = a^{\gamma_1} \cdot e^{-1\gamma_1} = (a - re) \cdot e^{-1} = a \cdot e^{-1} - r = q - r$ . The other two entries of  $u_1^{\gamma_1}$  are computed to be  $(q^t)^{\gamma_1} = q^t - r^t$ , and  $\frac{1}{2}(qq^t)^{\gamma_1} = \frac{1}{2}(qq^t + rr^t) - rq^t$ .

The  $\mathcal{K}_1$ -transformations of  $u_1$  is *dictated* by the  $\mathcal{K}_1$ -gauge transformations of the gauge potentials i.e. the entries of the Cartan connection matrix. It is thus remarkable that they precisely allow  $u_1$  to be truly a dressing field, as can be seen in matrix notation,

$$u_1^{\gamma_1} = \gamma_1^{-1} u_1 = \begin{pmatrix} 1 & -r & \frac{1}{2} r r^t \\ 0 & \mathbb{1} & -r^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q & \frac{1}{2} q q^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q - r & \frac{1}{2} (q q^t + r r^t) - r q^t \\ 0 & \mathbb{1} & q^t - r^t \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.4)$$



It is worthwhile to notice, on the contrary, that for  $\gamma_1, \gamma'_1 \in \mathcal{K}_1$ , a simple matrix calculation shows that  $\gamma'_1{}^{\gamma_1} := \gamma_1^{-1} \gamma'_1 \gamma_1 = \gamma'_1$ , since  $K_1$  is abelian.

Having now at our disposal the dressing field  $u_1$ , we can proceed to the dressing of the Cartan connection, in accordance with (2.3). One readily computes

$$\varpi_1 := \varpi^{u_1} = u_1^{-1} \varpi u_1 + u_1^{-1} du_1$$

$$\begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix} = \begin{pmatrix} a - q\theta & (a - q\theta)q + \alpha - qA + \frac{1}{2}qq^t\theta^t + dq & 0 \\ \theta & \theta q + A - q^t\theta^t & (\text{entry } (1,2))^t \\ 0 & \theta^t & \theta^t q^t - a \end{pmatrix}, \quad (3.5)$$

where, by construction,  $a_1 = a - q\theta \equiv 0$ . Likewise, for the dressed curvature,

$$\Omega_1 := \Omega^{u_1} = u_1^{-1} \Omega u_1$$

$$\begin{pmatrix} f_1 & \Pi_1 & 0 \\ \Theta_1 & F_1 & \Pi_1^t \\ 0 & \Theta_1^t & -f_1 \end{pmatrix} = \begin{pmatrix} f - q\Theta & \Pi - qF_1 + fq - \frac{1}{2}qq^t\Theta^t & 0 \\ \Theta & \Theta q + F - q^t\Theta^t & (\text{entry } (1,2))^t \\ 0 & \Theta^t & \Theta^t q^t - f \end{pmatrix}. \quad (3.6)$$

By construction,  $\varpi_1$  and  $\Omega_1$  are  $\mathcal{K}_1$ -gauge invariant composite fields. This means in particular that the expression of  $\varpi_1$  is invariant, that is  $a_1 = 0 = a_1^{\gamma_1}$ . It is worthwhile to notice that in [15] the condition  $a \equiv 0$  is considered as a gauge fixing, named “natural gauge” in [32], while it actually emerges through a dressing. Moreover in [33], by using an argument based on a specific Yang-Mills action, the Weyl potential  $a$  is set to zero while our approach is model independent.

Let us now study the behaviour of the composite field  $\varpi_1$  under Lorentz transformations,  $\text{SO}(1, m-1) \subset K_0$ , since any element  $\gamma_0 \in \mathcal{K}_0$  is factorized as

$$\gamma_0 = WS := \begin{pmatrix} z & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.7)$$

where  $z \in W$  and  $S \in \text{SO}(1, m-1)$  –with  $S^T \eta S = \eta$ – has been identified with its matrix-block representation. The action of the Weyl subgroup  $W$  is treated separately in appendix A.

By its very construction,  $\varpi_1^S$  depends on the Lorentz gauge transformations of the Cartan connection  $\varpi$  (set  $z = 1$  in (3.2))

$$\varpi^S = S^{-1} \varpi S + S^{-1} dS = \begin{pmatrix} a & \alpha S & 0 \\ S^{-1} \theta & S^{-1} A S + S^{-1} dS & S^{-1} \alpha^t \\ 0 & \theta^t S & -a \end{pmatrix}.$$

Since  $\theta^S = S^{-1} \theta = e^S \cdot dx = S^{-1} e \cdot dx$ , we get  $q^S = a^S \cdot (e^{-1})^S = a \cdot (e^{-1}) S = qS$ ,  $(q^S)^t = (qS)^t = S^{-1} q^t$  and thus  $(qq^t)^S = qq^t$ . Hence, the transformed  $u_1^S$  of  $u_1$  under the Lorentz group relies on those of the Cartan connection entries. Once more, it is remarkable

that in matrix notation it can be recast as

$$u_1^S = S^{-1}u_1S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & qS & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & S^{-1}q^t \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.8)$$

The latter shows that the  $K_1$ -valued dressing field  $u_1$  is subject to a Lorentz gauge-like transformation. Therefore, according to a calculation similar to (2.2), the composite fields  $\varpi_1$  and  $\Omega_1$  are Lorentz gauge transformed as

$$\varpi_1^S = S^{-1}\varpi_1S + S^{-1}dS, \quad \Omega_1^S = S^{-1}\Omega_1S. \quad (3.9)$$

This allows to conclude that the entries of  $\varpi_1$  and  $\Omega_1$  are *true* SO-gauge fields. This means that  $A_1$  is the true Lorentz/spin connection with curvature  $R_1 := dA_1 + A_1^2$ . To sum up,  $\varpi_1$  and  $\Omega_1$  are  $K_1$ -invariant composite fields but still remain SO-gauge fields.

Furthermore, if  $\varpi$  is chosen to be the *normal* conformal Cartan connection (3.1), then  $\varpi_1$  satisfies similar normality conditions:

$$\Theta_1 = 0, \quad \text{Ric}(F_1) = 0. \quad (3.10)$$

Indeed from (3.6) one has  $\Theta_1 = \Theta = 0$  and  $\text{Ric}(F_1) = \text{Ric}(\Theta q + F - q^t\Theta^t) = \text{Ric}(F) = 0$ . Furthermore, the trace-free condition  $f_1 = f - q\Theta = 0$  is obvious. We thus get

$$\varpi_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & \Pi_1 & 0 \\ 0 & F_1 & \Pi_1^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\alpha_1 + \alpha_1 A_1 & 0 \\ 0 & R_1 + \theta\alpha_1 + \alpha_1^t\theta^t & d\alpha_1^t + A_1\alpha_1^t \\ 0 & 0 & 0 \end{pmatrix} \quad (3.11)$$

where  $A_1$  is still the *Lorentz/spin-connection*,  $\alpha_1$  may be referred to as the *Schouten* 1-form (by solving in  $\alpha_1$  the equation  $\text{Ric}(F_1) = 0$ ),  $R_1$  is the *Riemann curvature* 2-form,  $F_1$  may be called the *Weyl curvature* 2-form, and finally,  $\Pi_1$  may be named the *Cotton* 2-form, that is the covariant differential of the Schouten 1-form with respect to the spin-connection  $A_1$ .

The SO-gauge fields  $\varpi_1$  and  $\Omega_1$  (normal or not) are associated with the *first-order structure*  $\mathcal{P}_0 := \mathcal{P}(\mathcal{M}, K_0) = CO(\mathcal{M})$ , since  $SO \subset K_0$ . It then makes sense to ask whether the Lorentz gauge symmetry can be neutralized by finding an adequate *second* dressing field. This is indeed possible as shown in the following.

**Second reduction.** We want to neutralize the Lorentz subgroup of  $K_0$ , leaving only the abelian Weyl subgroup as the final residual gauge symmetry. The suitable dressing field is extracted from  $\varpi^S$ , or  $\varpi_1^S$ . Indeed, we have  $\theta^S = S^{-1}\theta$  which provides the transformation law for the vielbein  $e \in GL_m(\mathbb{R})$ ,  $e^S = S^{-1}e$ . Hence, one may define the local map

$$u_0 : \mathcal{U} \rightarrow GL_{m+2}(\mathbb{R}) \supset K_0, \quad \text{with matrix form,} \quad u_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{such that}$$

$$u_0^S = S^{-1}u_0 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^S & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1}e & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.12)$$

Note that such a field  $u_0$  is valued in a larger group than the one to be erased,  $G' \simeq GL_m \supset SO = H'$ . Moreover, from  $\varpi^{\gamma_1}$ , see (3.3), one extracts

$$\theta^{\gamma_1} = \theta \quad \rightarrow \quad e^{\gamma_1} = e, \quad \text{which implies} \quad u_0^{\gamma_1} = u_0. \quad (3.13)$$

Eqs. (3.12) and (3.13) secure that  $\varpi_1$  can be dressed by  $u_0$ . This gives rise to a new composite field which is inert under both the  $\mathcal{K}_1$ - and SO-actions:

$$\begin{aligned} \varpi_0 &:= \varpi_1^{u_0} = u_0^{-1} \varpi_1 u_0 + u_0^{-1} du_0 \\ &=: \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1} \cdot P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 e & 0 \\ e^{-1} \theta & e^{-1} A_1 e + e^{-1} de & e^{-1} \alpha_1^t \\ 0 & \theta^t e & 0 \end{pmatrix}, \end{aligned} \quad (3.14)$$

where  $g$  is the metric on  $\mathcal{M}$  induced by the Cartan connection through  $e^T \eta e = g$ . In the last matrix equality,  $\Gamma := e^{-1} A_1 e + e^{-1} de$  and  $P := \alpha_1 e$  are *definitions*, the other entries are directly obtained from the calculation. In great detail

$$\begin{aligned} e^{-1} \theta &= e^{-1} e \cdot dx = \delta \cdot dx = dx, \\ \theta^t e &= \theta^T \eta e = dx^T \cdot e^T \eta e = dx^T \cdot g, \\ e^{-1} \alpha_1^t &= e^{-1} \eta^{-1} \alpha_1^T = g^{-1} \cdot e^T \alpha_1^T = g^{-1} \cdot (\alpha_1 e)^T = g^{-1} \cdot P^T. \end{aligned}$$

In components,  $\varpi_0$  thus reads

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta_\mu^\rho & \Gamma^\rho_{\mu\nu} & g^{\rho\lambda} P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \quad (3.15)$$

Actually, the transformation under coordinate changes of  $\varpi_0$ , in part due to  $u_0$ , allows to identify  $\Gamma$  as a linear connection 1-form. It turns out that  $\varpi_0$  is parametrized by geometric objects on  $\mathcal{M}$ . The curvature associated to  $\varpi_0$  is the following  $\mathcal{K}_1$ - and SO-invariant composite field

$$\Omega_0 := \Omega_1^{u_0} = u_0^{-1} \Omega_1 u_0 = \begin{pmatrix} f_1 & \Pi_1 e & 0 \\ e^{-1} \Theta & e^{-1} F_1 e & e^{-1} \Pi^t \\ 0 & \Theta^t e & -f_1 \end{pmatrix} =: \begin{pmatrix} f_0 & C & 0 \\ T & W & C^t \\ 0 & T^t & -f_0 \end{pmatrix}. \quad (3.16)$$

Instead, by computing  $\Omega_0 = d\varpi_0 + \varpi_0^2$  directly, the above matrix explicitly reads

$$\begin{pmatrix} f_0 & C & 0 \\ T & W & C^t \\ 0 & T^t & -f_0 \end{pmatrix} = \begin{pmatrix} P \wedge dx & dP + P \wedge \Gamma & 0 \\ \Gamma \wedge dx & R + dx \wedge P + g^{-1} \cdot P^T \wedge dx^T \cdot g & \nabla g^{-1} \wedge P^T + g^{-1} \cdot C^T \\ 0 & -dx^T \wedge (\nabla g + \Gamma^T \cdot g) & dx^T \wedge P^T \end{pmatrix}, \quad (3.17)$$

where  $R = d\Gamma + \Gamma^2$  is the curvature of the linear connection  $\Gamma$  on  $U$ , and  $\nabla g = dg - \Gamma^T g - g \Gamma$  is the covariant derivative of the metric with respect to the linear connection. The metricity

condition  $\nabla g = 0$  is automatically satisfied as can be checked due to the fact that  $A_1$  is  $\mathfrak{so}$ -valued. Expressing (3.17) in components reads

$$\begin{aligned}\Omega_0 &= \frac{1}{2} \begin{pmatrix} (f_0)_{\mu\sigma} & C_{\nu,\mu\sigma} & 0 \\ T_{\mu\sigma}^\rho & W_{\nu,\mu\sigma}^\rho & C_{\mu\sigma}^\rho \\ 0 & T_{\nu,\mu\sigma} & -(f_0)_{\mu\sigma} \end{pmatrix} dx^\mu \wedge dx^\sigma \\ &= \begin{pmatrix} P_{[\mu,\sigma]} & \partial_{[\mu} P_{\sigma],\nu} + P_{[\mu,\lambda} \Gamma_{\sigma]\nu}^\lambda & 0 \\ \Gamma_{[\mu,\sigma]}^\rho & \frac{1}{2} R_{\nu,\mu\sigma}^\rho + \delta_{[\mu}^\rho P_{\sigma],\nu} + P_{[\mu,\lambda} g^{\lambda\rho} g_{\sigma]\nu} & \nabla_{[\mu} g^{\rho\lambda} P_{\lambda,\sigma]} + g^{\rho\lambda} \frac{1}{2} C_{\lambda,\mu\sigma} \\ 0 & \nabla_{[\mu} g_{\sigma]\nu} + \Gamma_{[\mu,\sigma]}^\lambda g_{\lambda\nu} & -P_{[\mu,\sigma]} \end{pmatrix} dx^\mu \wedge dx^\sigma.\end{aligned}\quad (3.18)$$

Restricting ourselves to the *normal* case (3.1),  $\varpi_0$  provides the so-called *Riemannian parametrization of the normal conformal Cartan connection* [10]. The normality conditions on  $\varpi_0$  defined by:

$$T = 0, \quad \text{Ric}(W) = 0.$$

are fulfilled. Indeed, from both (3.16) and (3.18) one has

$$T = e^{-1} \Theta_1 = 0, \quad \text{and} \quad \text{Ric}(W) := \text{Ric}(e^{-1} F_1 e) = \text{Ric}(e^{-1} F e) = \text{Ric}(F) = 0, \quad (3.19)$$

and hence  $f_0 = f_1 = P \wedge dx = 0$ .

Accordingly,  $T = 0$  implies the symmetry of  $\Gamma$  in its lower indices, and it can be shown in the usual way that  $\Gamma$  can be expressed as a function of  $g$  in order to get the *Levi-Civita connection* on  $U$ . In addition, the condition  $f_0 = 0$  renders  $P_{\mu\sigma}$  symmetric which is nothing but the so-called *Schouten tensor*, so that  $C_{\nu,\mu\sigma} = \nabla_\mu P_{\sigma\nu}$  is the *Cotton tensor* and  $W_{\nu,\mu\sigma}^\rho$  is the *Weyl tensor*. To sum up, in the *normal* case we have,

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta_\mu^\rho & \Gamma_{\mu\nu}^\rho & g^{\rho\lambda} P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \quad \text{with} \quad P_{\mu\nu} = \frac{-1}{(m-2)} \left( R_{\mu\nu} - \frac{R}{2(m-1)} g_{\mu\nu} \right), \quad (3.20)$$

$$\Omega_0 = \frac{1}{2} \begin{pmatrix} 0 & C_{\nu,\mu\sigma} & 0 \\ 0 & W_{\nu,\mu\sigma}^\rho & g^{\rho\lambda} C_{\lambda,\mu\sigma} \\ 0 & 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma. \quad (3.21)$$

Formulae (3.20) and (3.21) are known as the Riemannian parametrization of the normal conformal Cartan connection. In full generality, without assuming the normality conditions, the composite field given in (3.14) thus provides a more general parametrization of the conformal Cartan connection.

The composite fields  $\varpi_0$  and  $\Omega_0$  (normal or not) are now associated to the Weyl bundle  $\mathcal{P}_W$ . Remark that this step of reducing  $\mathcal{P}_0$ , with structure group  $K_0 = W \times \text{SO} \simeq \text{CO}(1, m-1)$ , to  $\mathcal{P}_W$  with abelian structure group  $W$ , is quite analogous to the case of the electroweak sector of the Standard Model where the initial bundle with structure group  $\text{SU}(2) \times \text{U}(1)$  was reduced to a subbundle with abelian structure group  $\text{U}(1)$ ; see [19, 24].

**Two steps in one: the compatibility conditions.** It is possible to go from  $(\varpi, \Omega)$  to  $(\varpi_0, \Omega_0)$  in a single step, because the dressing fields obey necessary compatibility conditions which are the last two equations in the following collected set

$$\begin{aligned} u_1^{\gamma_1} &= \gamma_1^{-1} u_1, & u_0^S &= S^{-1} u_0 && \text{equation (3.4) and (3.12) respectively.} \\ u_1^S &= S^{-1} u_1 S, & u_0^{\gamma_1} &= u_0 && \text{equation (3.8) and (3.13).} \end{aligned}$$

These imply,

$$\begin{aligned} (u_1 u_0)^{\gamma_1} &= u_1^{\gamma_1} u_0^{\gamma_1} = \gamma_1^{-1} u_1 u_0 = \gamma_1^{-1} (u_1 u_0), \\ (u_1 u_0)^S &= u_1^S u_0^S = S^{-1} u_1 S S^{-1} u_0 = S^{-1} (u_1 u_0), \end{aligned}$$

which show that the double dressing field  $u_1 u_0$  is a dressing for the groups  $\mathcal{K}_1$  and  $\text{SO}$  respectively. The above two equations say more, since they imply (in full detail),

$$\begin{aligned} (u_1 u_0)^{S\gamma_1} &= \left( (u_1 u_0)^S \right)^{\gamma_1} = \left( S^{-1} (u_1 u_0) \right)^{\gamma_1} = (S^{\gamma_1})^{-1} (u_1 u_0)^{\gamma_1} \\ &= \gamma_1^{-1} S^{-1} \gamma_1 \gamma_1^{-1} (u_1 u_0) = \gamma_1^{-1} S^{-1} (u_1 u_0) = (S\gamma_1)^{-1} (u_1 u_0), \end{aligned}$$

so that  $u_1 u_0$  turns out to be a dressing field for the whole gauge subgroup  $\text{SO } \mathcal{K}_1$ .

To sum up, had we guessed the sole composite dressing field  $u := u_1 u_0 : \mathcal{U} \rightarrow K_1 GL_{m+2}(\mathbb{R})$ , we could have directly dressed the Cartan connection and its curvature in order to get at once the composite fields  $\varpi_0 := \varpi^u$  and  $\Omega_0 := \Omega^u$ , constructed above.

Since  $\varpi_0$  and  $\Omega_0$  are associated to the *Weyl bundle*  $\mathcal{P}_W$ , a *residual Weyl gauge symmetry* is expected to hold. The next section addresses this issue.

### 3.3 The residual Weyl symmetry

The goal of our analysis is to obtain the transformations of the final composite fields  $\varpi_0$  (eq. (3.14)) and  $\Omega_0$  (eq. (3.17)) under the residual Weyl symmetry. These residual transformations arise from the *combined* transformations under the Weyl group of the Cartan connection  $\varpi$  and its curvature  $\Omega$  on the one hand, and, of the two dressing fields, on the other hand. Indeed, one is led to define the Weyl transformation of the fields as

$$\varpi_0^W := (\varpi^{u_1 u_0})^W = (\varpi^W)^{u_1^W u_0^W}, \quad \text{and} \quad \Omega_0^W := (\Omega^{u_1 u_0})^W = (\Omega^W)^{u_1^W u_0^W}, \quad (3.22)$$

where  $W$  was defined in (3.7). The Weyl transformation of the Cartan connection is,

$$\varpi^W = W^{-1} \varpi W + W^{-1} dW = \begin{pmatrix} a + z^{-1} dz & z^{-1} \alpha & 0 \\ z\theta & A & \alpha^t z^{-1} \\ 0 & z\theta^t & -a + z dz^{-1} \end{pmatrix}, \quad (3.23)$$

from which we can see that  $\theta^W = z\theta \rightarrow e^W = ze$ . Turning this into a matrix notation, one has

$$u_0^W := \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^W & 0 \\ 0 & 0 & 1 \end{pmatrix} = \widetilde{W} u_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ze & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.24)$$

where  $\widetilde{W}$  is thus *another matrix representation* of the Weyl group, adapted to the dressing field  $u_0$ , different than the initial representation  $W$ .

The Weyl tranformation of the dressing  $u_1$  stems from (3.23):  $a^W = a + z^{-1}dz$ , where  $a$  is a 1-form, so that in covector notation  $a^W = a + z^{-1}\partial z =: a + \zeta$ , where  $a$  now stands for the scalar coefficients of the 1-form  $a := a \cdot dx$ . Then, given  $u_1 \sim q := a \cdot e^{-1}$ , one computes

$$q^W := a^W \cdot (e^{-1})^W = (a + \zeta) \cdot z^{-1}e^{-1} = z^{-1}(q + \zeta \cdot e^{-1}), \quad (3.25)$$

likewise,  $(q^t)^W = (q^W)^t$ . One readily verifies  $(qq^t)^W = q^W(q^W)^t$ . These transformation laws suggest that the abelian composition law in the subgroup  $K_1$  must enter into the game. Having this point in mind, a rather tricky matrix expression for the transformed field  $u_1^W$  is found out

$$\begin{aligned} u_1^W &= \begin{pmatrix} 1 & q^W & \frac{1}{2}q^W(q^W)^t \\ 0 & \mathbb{1} & (q^W)^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z^{-1}(q + \zeta \cdot e^{-1}) & \frac{1}{2}z^{-2}(q + \zeta \cdot e^{-1})(q + \zeta \cdot e^{-1})^t \\ 0 & \mathbb{1} & z^{-1}(q + \zeta \cdot e^{-1})^t \\ 0 & 0 & 1 \end{pmatrix} \quad (3.26) \\ &= W^{-1}u_1k_1W = W^{-1}k_1u_1W := W^{-1}u_1 \begin{pmatrix} 1 & \zeta \cdot e^{-1} & \frac{1}{2}\zeta \cdot e^{-1}(\zeta \cdot e^{-1})^t \\ 0 & \mathbb{1} & (\zeta \cdot e^{-1})^t \\ 0 & 0 & 1 \end{pmatrix} W \end{aligned}$$

which suggests to proceed as after (3.8) in order to get the Weyl transformations of the composite fields.

**Residual Weyl gauge symmetry of  $\varpi_0$ .** In virtue of the definition (3.22), for the composite field  $\varpi_0$  we have,

$$\begin{aligned} \varpi_0^W &:= (\varpi^W)^{u_1^W u_0^W} = (\varpi^W)^{W^{-1}u_1k_1W\widetilde{W}u_0} \\ &= \varpi^{u_1k_1u_0W\widetilde{W}} = \varpi^{u_1u_0u_0^{-1}k_1u_0W\widetilde{W}} = \varpi_0^{u_0^{-1}k_1u_0W\widetilde{W}} \\ &=: \bar{W}^{-1}\varpi_0\bar{W} + \bar{W}^{-1}d\bar{W}. \end{aligned}$$

where we have used the fact that  $[W\widetilde{W}, u_0] = 0$  and defined the  $z$ -dependent matrix

$$\bar{W} := u_0^{-1}k_1u_0W\widetilde{W} = \begin{pmatrix} z & z\zeta & \frac{1}{2}z^{-1}\zeta \cdot g^{-1} \cdot \zeta^T \\ 0 & z\delta & z^{-1}g^{-1} \cdot \zeta^T \\ 0 & 0 & z^{-1} \end{pmatrix}.$$

A straightforward calculation yields

$$\begin{aligned} &\begin{pmatrix} 0 & P^W & 0 \\ dx^W & \Gamma^W & (g^{-1}P^T)^W \\ 0 & (dx^T \cdot g)^W & 0 \end{pmatrix} = \\ &\begin{pmatrix} 0 & P + \nabla\zeta - \zeta \cdot dx\zeta + \frac{1}{2}\zeta \cdot g^{-1} \cdot \zeta^T dx^T \cdot g & 0 \\ dx & \Gamma + z^{-1}dz\delta + \zeta dx - g^{-1} \cdot \zeta^T dx^T \cdot g & z^{-2}g^{-1} \cdot (\text{entry } (1,2))^T \\ 0 & dx^T \cdot z^2g & 0 \end{pmatrix}. \quad (3.27) \end{aligned}$$

Let us elaborate on this result. Entry (2, 1) gives  $dx^W = dx$ , which is the obvious invariance of the coordinate chart under a Weyl rescaling. Entry (3, 2) gives,

$$(dx^T \cdot g)^W = dx^T \cdot g^W = dx^T \cdot (z^2 g), \quad \text{in components} \quad (g_{\mu\nu})^W = z^2 g_{\mu\nu}, \quad (3.28)$$

which is the *Weyl rescaling of the metric tensor* under the conformal factor  $z^2$ . Entry (2, 2) gives in components

$$(\Gamma^\rho_{\mu\nu})^W = \Gamma^\rho_{\mu\nu} + \delta_\nu^\rho \zeta_\mu + \delta_\mu^\rho \zeta_\nu - g^{\rho\lambda} \zeta_\lambda g_{\mu\nu}, \quad (3.29)$$

with  $\zeta_\mu := z^{-1} \partial_\mu z$ . Entry (1, 2) gives in components,

$$(P_{\mu\nu})^W = P_{\mu\nu} + \nabla_\mu \zeta_\nu - \zeta_\mu \zeta_\nu + \frac{1}{2} \zeta_\lambda \zeta^\lambda g_{\mu\nu}. \quad (3.30)$$

with  $\zeta^\lambda := g^{\lambda\alpha} \zeta_\alpha$ . Entry (2, 3) gives in components,

$$(g^{\rho\lambda} P_{\lambda\mu})^W = z^{-2} g^{\rho\lambda} (P_{\lambda\mu} + \nabla_\mu \zeta_\lambda - \zeta_\lambda \zeta_\mu + \frac{1}{2} g_{\lambda\mu} \zeta_\alpha \zeta^\alpha),$$

which is redundant with (3.28) and (3.30) respectively.

**Remark.** Equations (3.29) and (3.30) look like the familiar conformal transformations of the Christoffel symbols and of the Schouten tensor. Notice however that in this framework, the metricity condition  $\nabla g = 0$  being guaranteed,  $\Gamma$  reduces to the Levi-Civita connection in the *normal* case ( $T = 0$ ) *only*. We will return to that specific case down below. However, the above calculations hold even without restricting ourselves to this assumption. We then obtain at once the Weyl variation of both the symmetric *and* anti-symmetric parts of  $\Gamma$  and  $P$ . Explicitly, decomposing  $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{[\mu\nu]} + \Gamma^\rho_{(\mu\nu)}$  and  $P_{\mu\nu} = P_{[\mu\nu]} + P_{(\mu\nu)}$ , then for the Christoffel symbols of the linear connection

$$(\Gamma^\rho_{[\mu\nu]})^W = \Gamma^\rho_{[\mu\nu]}, \quad \text{and} \quad (\Gamma^\rho_{(\mu\nu)})^W = \Gamma^\rho_{(\mu\nu)} + \delta_\nu^\rho \zeta_\mu + \delta_\mu^\rho \zeta_\nu - g^{\rho\lambda} \zeta_\lambda g_{\mu\nu}, \quad (3.31)$$

while for the coefficients of the 1-form  $P$  one gets

$$\begin{aligned} (P_{[\mu\nu]})^W &= P_{[\mu\nu]} - \zeta_\lambda \Gamma^\lambda_{[\mu\nu]} \\ \text{and} \quad (P_{(\mu\nu)})^W &= P_{(\mu\nu)} + \partial_{(\mu} \zeta_{\nu)} - \zeta_\lambda \Gamma^\lambda_{(\mu\nu)} - \zeta_\mu \zeta_\nu + \frac{1}{2} \zeta_\lambda \zeta^\lambda g_{\mu\nu}, \end{aligned} \quad (3.32)$$

where  $\partial_{(\mu} \zeta_{\nu)} = \partial_\mu \zeta_\nu$  from the very definition of  $\zeta$ . The two equalities concerning the symmetric parts are nothing but the *transformations of the Christoffel symbols and of the Schouten tensor under Weyl rescaling of the metric*.

Thus, with (3.29) and (3.30), not only do we recover classical results in a much more effective way, but we have more: we do not need to assume a priori that  $\Gamma$  and  $P$  are functions of the metric  $g$ , as it is usually the case when one works with the Levi-Civita connection. It is pretty noticeable to find out these general Weyl transformations as a ‘top-down’ process.

**Residual Weyl gauge symmetry of  $\Omega_0$ .** Similarly, for the composite field  $\Omega_0$  (eq. (3.17)), formula (3.22) gives

$$\begin{aligned}\Omega_0^W &:= (\Omega^W)^{u_1^W u_0^W} = (\Omega^W)^{W^{-1} u_1 k_1 W \tilde{W} u_0} = \Omega^{u_1 u_0 u_0^{-1} k_1 u_0 W \tilde{W}} =: \bar{W}^{-1} \Omega_0 \bar{W}, \\ \begin{pmatrix} f_0^W & C^W & 0 \\ T^W & W^W & C^{tW} \\ 0 & T^{tW} & -f_0^W \end{pmatrix} &= \\ \begin{pmatrix} f_0 - \zeta \cdot T & C - \zeta \cdot W + (f_0 - \zeta \cdot T)\zeta + \frac{1}{2}\zeta \cdot g^{-1} \cdot \zeta^T T^T \cdot g & 0 \\ T & W + T\zeta - g^{-1} \cdot \zeta^T T^T \cdot g & z^{-2} g^{-1} (\text{entry } (1, 2))^T \\ 0 & T^T \cdot z^2 g & (\zeta \cdot T)^T - f_0 \end{pmatrix}. \end{aligned} \quad (3.33)$$

In components, entries (1, 1) and (3, 3) give

$$(2P_{[\mu, \sigma]})^W = 2P_{[\mu, \sigma]} - \zeta_\lambda T^\lambda_{\mu\sigma}, \quad (3.34)$$

which just reproduces the first equation in (3.32) above, since  $\frac{1}{2}T^\lambda_{\mu\sigma} = \Gamma^\lambda_{[\mu, \sigma]}$ . Entry (2, 1) gives,

$$(T^\rho_{\mu\sigma})^W = T^\rho_{\mu\sigma} \quad (3.35)$$

which reproduces the first equation in (3.31).

Entry (3, 2) is redundant with (3.35) and (3.28). Entry (2, 2) gives,

$$(W^\rho_{\nu, \mu\sigma})^W = W^\rho_{\nu, \mu\sigma} + T^\rho_{\mu\sigma} \zeta_\nu - g^{\rho\lambda} \zeta_\lambda T_{\mu\sigma}{}^\alpha g_{\alpha\nu} \quad (3.36)$$

Entry (1, 2) leads to

$$(C_{\nu, \mu\sigma})^W = C_{\nu, \mu\sigma} - \zeta_\lambda W^\lambda_{\nu, \mu\sigma} + \zeta_\lambda \delta_\nu^\lambda f_{\mu\sigma} + \frac{1}{2}(\zeta_\lambda g^{\lambda\alpha} \zeta_\alpha) T_{\mu\sigma}{}^\beta g_{\beta\nu} - \zeta_\lambda T^\lambda_{\mu\sigma} \zeta_\nu. \quad (3.37)$$

At last, entry (2, 3) is redundant with (3.28) and (3.37).

**Residual Weyl gauge freedom of  $\varpi_0$  and  $\Omega_0$  in the normal case.** It is now straightforward to specialize the above transformations to the case where the initial Cartan connection  $\varpi$  is normal. The dressed normal conformal Cartan connection  $\varpi_0$  has been given in (3.20), with curvature (3.21). It is readily seen from the Weyl variation  $\Omega_0^W$  given in (3.33) that the normality of  $\varpi_0$ , as stated by (3.19), is preserved under the action of the Weyl gauge group.

Formula (3.27) remains formally unchanged but  $\Gamma$  becomes the Levi-Civita connection and  $P$  is the Schouten tensor, while the Weyl variation of the curvature (3.33) reduces to

$$\Omega_0^W = \begin{pmatrix} 0 & C^W & 0 \\ 0 & W^W & C^{Wt} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C - \zeta \cdot W & 0 \\ 0 & W & z^{-2} g^{-1} (C - \zeta \cdot W)^T \\ 0 & 0 & 0 \end{pmatrix}.$$



The Weyl rescaling of the metric tensor is still given by (3.28). But (3.29) and (3.30) now expresses respectively the transformations of the Levi-Civita connection and of the Schouten tensor under Weyl rescaling. Equation (3.36) reduces to

$$(W^\rho{}_{\nu,\mu\sigma})^W = W^\rho{}_{\nu,\mu\sigma} \quad (3.38)$$

and is the well known *invariance of the Weyl tensor under Weyl rescaling*. Finally, (3.37) gives

$$(C_{\nu,\mu\sigma})^W = C_{\nu,\mu\sigma} - \zeta_\lambda W^\lambda{}_{\nu,\mu\sigma}, \quad (3.39)$$

which is the *transformation of the Cotton-York tensor under Weyl rescaling*.

## 4 The dressing field and the BRST framework

Since its inception in the mid 70's by Becchi, Rouet and Stora [21, 22] and by Tyutin [23], the BRST formalism has seeded considerable work, and has been generalized to a large class of theories and became a standard tool in the analysis of gauge field theories and their quantization. For further developments on the techniques, see for instance [34–39].

In the following, we give the minimal definition of the BRST algebra of a theory and show how the latter is modified through the dressing field method. Then we apply the construction to General Relativity (GR) and to the second order conformal structure.

### 4.1 Residual BRST algebra

Consider the bundle  $\mathcal{P}(\mathcal{M}, H)$  with a (local) connection  $A$  together with a section  $\psi$  of any associated bundle. Given the  $\mathfrak{h}$ -valued ghost  $v$ , the BRST algebra is,

$$sA = -Dv, \quad sF = [F, v], \quad s\psi = -\rho_*(v)\psi, \quad \text{and} \quad sv = -\frac{1}{2}[v, v],$$

with  $D = d + [\omega, \ ]$ , and  $F = dA + \frac{1}{2}[A, A]$ . The commutator  $[\alpha, \beta] = \alpha\beta - (-)^{|\alpha||\beta|}\beta\alpha$  is graded according to total degree which consists of the form and ghost degrees.  $\rho_*$  is the representation of the Lie algebra  $\mathfrak{h}$  on matter fields. The BRST operator,  $s$ , is nilpotent:  $s^2 = 0$ .

This can be compactly rewritten under the elegant algebraic equation, the so-called “Russian formula” [29] (also named “horizontal condition” [40, 41])

$$(d + s)(A + v) + \frac{1}{2}[A + v, A + v] = F, \quad (4.1)$$

according to an expansion with respect to the ghost degree. The sum  $A + v$  is named the “algebraic connection” [42].

As a first basic result, one states:

**Lemma 1** (Modified BRST algebra). *Let  $u : U \rightarrow G'$  be a field as in section 2 on which the action of the gauge group  $\mathcal{H}$  is not specified yet. Let*

$$\hat{A} := u^{-1}Au + u^{-1}du, \quad \hat{F} = u^{-1}Fu, \quad \text{and} \quad \hat{\psi} := \rho(u^{-1})\psi,$$

be the corresponding composite fields. Then there is a modified BRST algebra:

$$s\hat{A} = -\hat{D}\hat{v}, \quad s\hat{F} = [\hat{F}, \hat{v}], \quad s\hat{\psi} = -\rho_*(\hat{v})\hat{\psi}, \quad \text{and} \quad s\hat{v} = -\frac{1}{2}[\hat{v}, \hat{v}], \quad (4.2)$$

with new ghost given by

$$\hat{v} = u^{-1}vu + u^{-1}su. \quad (4.3)$$

This *composite ghost* is the central new variable. To the best of our knowledge, first appearances of such an object in specific examples can be found in [8, 43].

*Proof.* It is easily checked by expressing each variable of the initial BRST algebra as function of the composite variables and the dressing field. In the course of the checking, the explicit expression for the variation  $su$  is not required.  $\square$

Accordingly, we have the following

**Corollary 2.** *One can define a composite algebraic connection,*

$$\hat{A} + \hat{v} = u^{-1}(A + v)u + u^{-1}(d + s)u \quad (4.4)$$

*which, by virtue of the above modified BRST algebra, satisfied the modified Russian formula,*

$$(d + s)(\hat{A} + \hat{v}) + \frac{1}{2}[\hat{A} + \hat{v}, \hat{A} + \hat{v}] = \hat{F}. \quad (4.5)$$

**Three relevant possibilities.** The modified BRST algebra of Lemma 1 can take various presentations according to explicit expression of the *composite ghost* which is dictated by a given BRST transformation of the field  $u : U \rightarrow G'$ . In this respect, three cases will be considered.

First case. Suppose  $u$  is subject to a *gauge-like* finite  $\mathcal{H}$ -transformation, that is  $u^\gamma = \gamma^{-1}u\gamma$ , for  $\gamma \in \mathcal{H}$ . Then its BRST transformation, mimicking infinitesimal gauge transformations with the ghost  $v$  as  $\mathfrak{h}$ -valued parameter, is

$$su = [u, v]. \quad (4.6)$$

This implies that the ghost is kept unchanged  $\hat{v} = v$  and (4.2) reads

$$s\hat{A} = -\hat{D}v, \quad s\hat{F} = [\hat{F}, v], \quad s\hat{\psi} = -\rho_*(v)\hat{\psi}, \quad \text{and} \quad sv = -\frac{1}{2}[v, v]. \quad (4.7)$$

This is not a surprise since if  $u$  were a gauge element  $u \in \mathcal{H}$ , the fields  $\hat{A}$ ,  $\hat{F}$  and  $\hat{\psi}$  would actually be the  $\mathcal{H}$ -gauge transformed of  $A$ ,  $F$  and  $\psi$  respectively, satisfying the same BRST algebra.

Second case. Suppose that  $u$  is a *dressing field* for  $\mathcal{H}$  that is  $u^\gamma = \gamma^{-1}u$ , for  $\gamma \in \mathcal{H}$ . Then its BRST transformation is,

$$su = -vu. \quad (4.8)$$

This implies that the *composite ghost* vanishes  $\widehat{v} = 0$ . The  $\mathcal{H}$ -symmetry is thus annihilated and the *modified BRST algebra* (4.2) reduces to the *trivial algebra*,

$$s\widehat{A} = 0, \quad s\widehat{F} = 0, \quad \text{and} \quad s\widehat{\psi} = 0. \quad (4.9)$$

This expresses that  $\widehat{A}$ ,  $\widehat{F}$  and  $\widehat{\psi}$  are  $\mathcal{H}$ -*gauge invariants* fields. In that case, it is stressed that the composite fields coming out from the dressing field method ought to be good candidates for being “observables”.

Third case. Let  $H' \subset H$  be a Lie subgroup with Lie algebra  $\mathfrak{h}'$ . Suppose that  $u$  transforms according to  $u^{\gamma'} = \gamma'^{-1}u$  for  $\gamma' \in \mathcal{H}'$ . For the time being, we left unspecified the transformation  $u^{\gamma_0}$  for  $\gamma_0 \in \mathcal{H} \setminus \mathcal{H}'$ . Suppose we can find a  $\text{Ad}_H$ -invariant complement,  $\mathfrak{p}$ , to  $\mathfrak{h}'$  in  $\mathfrak{h}$ , so that  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{p}$ . The ghost splits according to  $v = v_{\mathfrak{h}} = v_{\mathfrak{h}'} + v_{\mathfrak{p}}$  and accordingly the BRST operator splits too as  $s = s_{\mathfrak{h}} = s_{\mathfrak{h}'} + s_{\mathfrak{p}}$  with

$$\begin{aligned} s^2 = 0 &\Leftrightarrow s_{\mathfrak{h}'}^2 = s_{\mathfrak{p}}^2 = s_{\mathfrak{h}'} s_{\mathfrak{p}} + s_{\mathfrak{p}} s_{\mathfrak{h}'} = 0, \\ sv = -\frac{1}{2}[v, v] &\Leftrightarrow s_{\mathfrak{h}'} v_{\mathfrak{h}'} = -\frac{1}{2}[v_{\mathfrak{h}'}, v_{\mathfrak{h}'}], \quad s_{\mathfrak{h}'} v_{\mathfrak{p}} = -[v_{\mathfrak{h}'}, v_{\mathfrak{p}}], \quad s_{\mathfrak{p}} v_{\mathfrak{p}} = -\frac{1}{2}[v_{\mathfrak{p}}, v_{\mathfrak{p}}]. \end{aligned} \quad (4.10)$$

The BRST transformation of the dressing field is then,

$$su = s_{\mathfrak{h}}u = s_{\mathfrak{h}'}u + s_{\mathfrak{p}}u = -v_{\mathfrak{h}'}u + s_{\mathfrak{p}}u, \quad (4.11)$$

where  $s_{\mathfrak{p}}u$  is left unspecified. This implies for the *composite ghost*,

$$\begin{aligned} \widehat{v} &= u^{-1}v_{\mathfrak{h}}u + u^{-1}s_{\mathfrak{h}}u = u^{-1}v_{\mathfrak{h}'}u + u^{-1}v_{\mathfrak{p}}u - u^{-1}v_{\mathfrak{h}'}u + u^{-1}s_{\mathfrak{p}}u \\ &= u^{-1}v_{\mathfrak{p}}u + u^{-1}s_{\mathfrak{p}}u =: \widehat{v}_{\mathfrak{p}}, \end{aligned} \quad (4.12)$$

showing that the whole  $\mathcal{H}'$  subsector has been neutralized. The composite ghost  $\widehat{v}_{\mathfrak{p}}$  encodes the residual gauge symmetry. Thus, the *modified BRST algebra* (4.2) reads

$$s\widehat{A} = -\widehat{D}\widehat{v}_{\mathfrak{p}}, \quad s\widehat{F} = [\widehat{F}, \widehat{v}_{\mathfrak{p}}], \quad s\widehat{\psi} = -\rho_*(\widehat{v}_{\mathfrak{p}})\widehat{\psi}, \quad \text{and} \quad s\widehat{v}_{\mathfrak{p}} = -\frac{1}{2}[\widehat{v}_{\mathfrak{p}}, \widehat{v}_{\mathfrak{p}}], \quad (4.13)$$

and gives the *infinitesimal residual  $\mathcal{H}/\mathcal{H}'$  gauge transformations* of the composite fields  $\widehat{A}$ ,  $\widehat{F}$  and  $\widehat{\psi}$ . This expresses the *reduction* of the BRST algebra we started with.

These three cases cover pertinent types of erasure (none, total, partial) of gauge symmetry. Let us apply these mechanisms to GR, on the one hand, and to the second order conformal structure, on the other hand.

## 4.2 Application to the geometry of General Relativity

The geometry underlying GR considered as a gauge theory, is a Cartan geometry  $(\mathcal{P}, \varpi)$ , where  $\mathcal{P}(\mathcal{M}, H) = \text{SO}(\mathcal{M})$  is a principal bundle with  $H = \text{SO}(1, m-1)$  the Lorentz group and  $\varpi \in \Omega^1(U, \mathfrak{g})$  is a (local) Cartan connection on  $U \subset \mathcal{M}$  with values in  $\mathfrak{g}$  the Lie algebra of the Poincaré group  $G = \text{SO} \ltimes \mathbb{R}^m$ . One has the matrix writing

$$\varpi = \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^a_{b,\mu} & e^a_{\mu} \\ 0 & 0 \end{pmatrix} dx^{\mu},$$

with  $A \in \Omega^1(\mathcal{U}, \mathfrak{h})$  the Lorentz connection and  $\theta \in \Omega^1(\mathcal{U}, \mathbb{R}^m)$  the vielbein 1-form. The greek indices are spacetime indices, while latin indices are “internal” (gauge)-Minkowski indices. The curvature is

$$\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] = d\varpi + \varpi^2 \quad \rightarrow \quad \begin{pmatrix} F & \Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} dA + A^2 & d\theta + A\theta \\ 0 & 0 \end{pmatrix},$$

with  $F$  the Riemann 2-form and  $\Theta$  the torsion 2-form. In this matrix notation, the Lorentz ghost reads  $v = \begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix}$  and the Lorentz BRST algebra is

$$\begin{aligned} s\varpi &= -Dv, & s\Omega &= [\Omega, v], & sv &= -\frac{1}{2}[v, v] = -v^2, \\ \begin{pmatrix} sA & s\theta \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -Dv_L & -v_L\theta \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} sF & s\Theta \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} [F, v_L] & -v_L\Theta \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} sv_L & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -v_L^2 & \theta \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Of course  $s^2 = 0$ . This matrix algebra handles the infinitesimal SO-gauge transformations of the variables of the theory.

As proposed in [19] the dressing field is the vielbein,  $u = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} : \mathcal{U} \rightarrow GL_m(\mathbb{R})$ . The corresponding composite fields express as

$$\hat{\varpi} = u^{-1}\varpi u + u^{-1}du = \begin{pmatrix} e^{-1}Ae + e^{-1}de & e^{-1}\theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix},$$

where  $\Gamma$  is a linear connection compatible with the metric defined by  $g = e^T\eta e$ , and

$$\hat{\Omega} = u^{-1}\Omega u = \hat{D}\hat{\varpi} = d\hat{\varpi} + \hat{\varpi}^2 = \begin{pmatrix} d\Gamma + \Gamma^2 & \Gamma \cdot dx \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} R & T \\ 0 & 0 \end{pmatrix},$$

where  $R$  is the Riemann tensor and  $T$  is the torsion tensor.

The key element of the modified BRST algebra is of course the composite ghost. To find its expression we only need to determine how the field  $u$  transforms under the action of the (initial) BRST operator. It is readily read from  $s\varpi$  above:  $s\theta = -v_L\theta \rightarrow se \cdot dx = -v_L e \cdot dx$ . Hence,

$$su = -vu \rightarrow \begin{pmatrix} se & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -v_L e & 0 \\ 0 & 0 \end{pmatrix},$$

which is of course the defining BRST transformation of a dressing field. Therefore the composite ghost vanishes by

$$\hat{v} = u^{-1}vu + u^{-1}su = u^{-1}vu + u^{-1}(-vu) = 0,$$

and we have the trivial modified BRST algebra,

$$s\hat{\varpi} = \begin{pmatrix} s\Gamma & sdx \\ 0 & 0 \end{pmatrix} = 0, \quad s\hat{\Omega} = \begin{pmatrix} sR & sT \\ 0 & 0 \end{pmatrix} = 0.$$

This expresses the invariance of the coordinate chart, and by construction the SO-gauge invariance of  $\Gamma$ ,  $R$  and  $T$ . The composite fields  $\widehat{\omega}$  and  $\widehat{\Omega}$  belong to the natural geometry of  $\mathcal{M}$ . They are blind to the initial Lorentz gauge symmetry since the latter has been fully neutralized by the dressing field  $u$ . GR illustrates case two discussed above. Let us now turn to the following less trivial example.

### 4.3 Application to the second order conformal structure

According to section 3.1, the structure group of the second order conformal structure  $\mathcal{P}(\mathcal{M}, H)$  is  $H = K_0 K_1 = (\text{SO} \times \text{W}) K_1$ . Turning the infinitesimal parameters  $(v_L, \epsilon, \iota)$  into Faddeev-Popov ghosts for the Lie algebra  $\mathfrak{co}(1, m-1) \oplus \mathbb{R}^{m*} = \mathfrak{so}(1, m-1) \oplus \mathbb{R} \oplus \mathbb{R}^{m*}$ , the matrix-wise ghost decomposes into symmetry sectors ( $L$  for Lorentz,  $W$  for Weyl and  $i$  for the inversions) as

$$v = \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & v_L & \iota^t \\ 0 & 0 & -\epsilon \end{pmatrix} = v_0 + v_i = v_L + v_i + v_W = \begin{pmatrix} 0 & \iota & 0 \\ 0 & v_L & \iota^t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}, \quad (4.14)$$

where the Lorentz ghost  $v_L$  is identified with its matrix block representation. The corresponding Slavnov operation splits accordingly as

$$s = s_0 + s_i = s_W + s_L + s_i,$$

and fulfils (4.10).

With the  $K_1$ -dressing field  $u_1 : \text{U} \rightarrow K_1$ , see (3.4), in addition to the composite fields  $\varpi_1 = \varpi^{u_1}$ ,  $\Omega_1 = \Omega^{u_1}$ , one constructs the first composite ghost

$$\widehat{v}_1 := v^{u_1} = u_1^{-1} v u_1 + u_1^{-1} s u_1 = u_1^{-1} (v_W + v_L + v_i) u_1 + u_1^{-1} (s_W + s_L + s_i) u_1.$$

Upon linearizing the finite transformations (3.4), (3.8) and (3.26), one obtains the following BRST variations of the dressing field  $u_1$

$$\begin{aligned} s_i u_1 &= -v_i u_1, \\ s_L u_1 &= [u_1, v_L], \end{aligned} \quad \text{and} \quad s_W u_1 = \begin{pmatrix} 0 & -\epsilon q + \partial \epsilon \cdot e^{-1} & -\epsilon q q^t + \partial \epsilon \cdot e^{-1} q^t \\ 0 & 0 & (-\epsilon q + \partial \epsilon \cdot e^{-1})^t \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.15)$$

which illustrate the three cases discussed in section 4.1. The first *composite ghost* is then

$$\begin{aligned} \widehat{v}_1 &= u_1^{-1} v_W u_1 + u_1^{-1} v_L u_1 + u_1^{-1} v_i u_1 + u_1^{-1} s_W u_1 + u_1^{-1} [u_1, v_L] - u_1^{-1} v_i u_1 \\ &= u_1^{-1} v_W u_1 + u_1^{-1} s_W u_1 + v_L \\ &= \begin{pmatrix} \epsilon & \partial \epsilon \cdot e^{-1} & 0 \\ 0 & v_L & (\partial \epsilon \cdot e^{-1})^t \\ 0 & 0 & -\epsilon \end{pmatrix}. \end{aligned} \quad (4.16)$$

In the course of the computation, the ghost  $v_i$  for the inversion sector has been killed by the dressing  $u_1$  so that the subalgebra corresponding to  $s_i$  is now trivial,

$$s_i \omega_1 = 0 \quad \text{and} \quad s_i \Omega_1 = 0, \quad (4.17)$$

and expresses the expected  $\mathcal{K}_1$ -invariance of the composite fields  $\varpi_1$  and  $\Omega_1$ . Thanks to the dressing  $u_1$ , the BRST operator  $s = s_0 + s_i = s_W + s_L + s_i$  reduces to  $s_0 = s_W + s_L$ , and the Lorentz subalgebra is kept unchanged,

$$s_L \varpi_1 = -D_1 v_L \quad \text{and} \quad s_L \Omega_1 = [\Omega_1, v_L],$$

where  $D_1 = d + [\varpi_1, \ ]$  is the covariant derivative with respect to  $\varpi_1$ . This means that  $\varpi_1$  and  $\Omega_1$  still behaves as connection and curvature under the Lorentz gauge group  $\text{SO}$ , as warranted by  $s_L u_1$  in (4.15). This is a hint, according to the example of GR treated in section 4.2, that a new dressing operation to neutralize the Lorentz symmetry can be performed.

**Second reduction and final BRST algebra.** With the dressing  $u_0: \text{U} \rightarrow GL_{m+2}(\mathbb{R}) \supset \text{SO}(1, m-1)$  given by (3.12), beside the composite fields  $\varpi_0 := \varpi_1^{u_0}$  and  $\Omega_0 := \Omega_1^{u_0}$  respectively given by (3.14) and (3.17), one construct the second composite ghost

$$\begin{aligned} \hat{v}_0 &= u_0^{-1} \hat{v}_1 u_0 + u_0^{-1} s_0 u_0 \\ &= u_0^{-1} (u_1^{-1} v_W u_1 + u_1^{-1} s_W u_1) u_0 + u_0^{-1} v_L u_0 + u_0^{-1} s_W u_0 + u_0^{-1} s_L u_0. \end{aligned}$$

The BRST transformations of  $u_0$  under  $s_W$  and  $s_L$  are respectively obtained by linearizing  $u_0^W$  (3.24) and  $u_0^S$  (3.12):

$$s_W u_0 = \tilde{\epsilon} u_0 \quad \rightarrow \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_W e & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_L u_0 = -v_L u_0 \quad (4.18)$$

The *final composite ghost* is then,

$$\begin{aligned} \hat{v}_0 &= u_0^{-1} (u_1^{-1} v_W u_1 + u_1^{-1} s_W u_1) u_0 + u_0^{-1} v_L u_0 + u_0^{-1} \tilde{\epsilon} u_0 + u_0^{-1} (-v_L u_0), \\ &= u_0^{-1} (u_1^{-1} v_W u_1 + u_1^{-1} s_W u_1) u_0 + \tilde{\epsilon} u_0^{-1} u_0, \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \partial \epsilon \cdot e^{-1} & 0 \\ 0 & 0 & \eta^{-1} (e^{-1})^T \cdot \partial \epsilon \\ 0 & 0 & -\epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} \epsilon & \partial \epsilon & 0 \\ 0 & \epsilon \delta & g^{-1} \partial \epsilon \\ 0 & 0 & -\epsilon \end{pmatrix} =: \hat{v}_W. \end{aligned} \quad (4.19)$$

Comparison with the first composite ghost (4.16) shows that the ghost  $v_L$  has been killed by the second dressing  $u_0$ , and the Lorentz subsymmetry corresponding to  $s_L$  is now trivial,

$$s_L \varpi_0 = 0 \quad \text{and} \quad s_L \Omega_0 = 0. \quad (4.20)$$

Furthermore the dressing  $u_0$  satisfies the compatibility condition  $s_i u_0 = 0$ , which is the infinitesimal version of (3.13), so that we also have

$$s_i \varpi_0 = 0 \quad \text{and} \quad s_i \Omega_0 = 0. \quad (4.21)$$

The triviality of these two subalgebras expresses the  $\mathcal{K}_1$ - and SO-invariance of the composite fields  $\varpi_0$  and  $\Omega_0$ . Their *infinitesimal residual Weyl gauge transformations* are given by the *final reduced BRST algebra*, with  $s_W^2 = 0$ ,

$$s_W \varpi_0 = -D_0 \hat{v}_W, \quad s_W \Omega_0 = [\Omega_0, \hat{v}_W], \quad \text{and} \quad s_W \hat{v}_W = -\hat{v}_W^2, \quad (4.22)$$

with a final composite ghost  $\hat{v}_W$  depending only on the Weyl ghost  $\epsilon$  and its first order derivatives.

**Two steps in one.** Mirroring the finite version given in section 3.2, we can reduce the initial BRST algebra with  $s = s_W + s_L + s_i$  to the final residual BRST algebra with  $s_W$  in a single step thanks to the dressing field  $u = u_1 u_0 : \mathcal{U} \rightarrow K_1 GL_{m+2}(\mathbb{R})$ . The corresponding composite ghost reads,

$$\hat{v} = u^{-1} v u + u^{-1} s u = u^{-1} (v_W + v_L + v_i) u + u^{-1} (s_W + s_L + s_i) u$$

One collects the BRST variations of the two dressing fields  $u_1$  and  $u_0$

$$s_i u_1 = -v_i u_1, \quad s_L u_0 = -v_L u_0, \quad \text{and} \quad s_L u_1 = [u_1, v_L], \quad s_i u_0 = 0,$$

the last two being *compatibility conditions*,<sup>2</sup> and proves  $(s_L + s_i)u = -(v_L + v_i)u$ . This means that  $u$  is a dressing under the subgroup  $\text{SO } \mathcal{K}_1$ . Hence, the composite ghost reduces to (4.19) since

$$\hat{v} = u^{-1} v_W u + u^{-1} s_W u = \hat{v}_W.$$

We see right away that  $v_L$  and  $v_i$  have been killed by the dressing  $u$  so that the corresponding subalgebras are trivial,

$$s_i \varpi_0 = 0 \quad \text{and} \quad s_i \Omega_0 = 0; \quad s_L \varpi_0 = 0 \quad \text{and} \quad s_L \Omega_0 = 0.$$

The triviality of these two subalgebras means that the composite fields  $\varpi_0$  and  $\Omega_0$  display only a residual Weyl gauge freedom handled by the *residual Weyl BRST algebra* given by (4.22).

To sum up, one has a dressing field

$$u := u_1 u_0 = \begin{pmatrix} 1 & q & \frac{1}{2} q q^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.23)$$

with  $u_1$  defined by  $a - q\theta = 0$  a gauge fixing like condition depending *locally* on entries of the gauge field  $\varpi$ . Upon writing

$$s u u^{-1} = \begin{pmatrix} 0 & -\iota & 0 \\ 0 & -v_L & -\iota^t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial \epsilon \cdot e^{-1} & -\epsilon q q^t \\ 0 & \epsilon \mathbb{1} & (\partial \epsilon \cdot e^{-1})^t - 2\epsilon q^t \\ 0 & 0 & 0 \end{pmatrix} =: -\ell + \varrho u^{-1}$$

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<sup>2</sup>See [20] for a general treatment in the BRST case.

the BRST variation reads  $su = -\ell u + \varrho$ , with  $\varrho = s_W u$ , see (4.11). The first ghost matrix  $-\ell = -v_L - v_i$  in the r.h.s. cancels out both the  $\mathcal{K}_1$  and Lorentz  $SO(1, m-1)$  actions according to

$$\widehat{v} = u^{-1}vu + u^{-1}su = u^{-1}(\ell + v_W)u + u^{-1}(-\ell u + \varrho) = u^{-1}v_W u + u^{-1}\varrho = \widehat{v}_W$$

as a realisation of formula (4.12).

**The residual Weyl BRST algebra: explicit results.** We restrict ourselves to the *normal* Cartan geometry. Let us recall (see (3.20) and (3.21))

$$\varpi_0 := \varpi^u = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1} \cdot P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_0 := \Omega^u = \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1} \cdot C^T \\ 0 & 0 & 0 \end{pmatrix}.$$

The residual BRST algebra given by (4.22) is straightforwardly computed thanks to the matrix form. The infinitesimal residual Weyl gauge transformation of the dressed *normal* conformal Cartan connection is,

$$\begin{aligned} s_W \varpi_0 &= -d\widehat{v}_W - \varpi_0 \widehat{v}_W - \widehat{v}_W \varpi_0 \\ &= \begin{pmatrix} -d\epsilon & -d(\partial\epsilon) & 0 \\ 0 & -d\epsilon\delta & -d(g^{-1} \cdot \partial\epsilon) \\ 0 & 0 & d\epsilon \end{pmatrix} - \begin{pmatrix} 0 & P \cdot \epsilon\delta & P g^{-1} \partial\epsilon \\ dx\epsilon & dx\partial\epsilon + \Gamma\epsilon\delta & \Gamma g^{-1} \partial\epsilon - g^{-1} P^T \epsilon \\ 0 & dx^T \cdot g \epsilon\delta & dx^T \cdot g g^{-1} \partial\epsilon \end{pmatrix} \\ &\quad - \begin{pmatrix} \partial\epsilon dx & \epsilon P + \partial\epsilon \Gamma & \partial\epsilon g^{-1} P^T \\ \epsilon\delta dx & \epsilon\delta \Gamma + g^{-1} d\epsilon dx^T \cdot g & \epsilon\delta g^{-1} \cdot P^T \\ 0 & -\epsilon dx^T \cdot g & 0 \end{pmatrix}. \end{aligned}$$

Reminding that  $\epsilon$  anticommutes with forms of odd degree, that  $d = dx \cdot \partial$  and using the metricity condition  $\nabla g^{-1} = dg^{-1} + g^{-1} \Gamma^T + \Gamma g = 0$  in the computation of entry (2, 3), we obtain

$$\begin{aligned} &\begin{pmatrix} 0 & s_W P & 0 \\ s_W dx & s_W \Gamma & s_W(g^{-1} \cdot P^T) \\ 0 & s_W(g \cdot dx) & 0 \end{pmatrix} = \\ &\begin{pmatrix} 0 & -\nabla \partial\epsilon & 0 \\ 0 & -d\epsilon\delta - dx\partial\epsilon - g^{-1} d\epsilon dx^T \cdot g & -g^{-1} \cdot (\nabla \partial\epsilon)^T - 2\epsilon g^{-1} \cdot P^T \\ 0 & 2\epsilon dx^T \cdot g & 0 \end{pmatrix}, \end{aligned}$$

where,  $\nabla \partial\epsilon = d(\partial\epsilon) + \partial\epsilon \cdot \Gamma$ , is the covariant derivative. Let us detail each entry in components. Entry (2,1) is  $s_W dx^\mu = 0$  and expresses the invariance of the coordinate chart. This will be of constant use for the other entries. Entry (3, 2) is then,

$$s_W g_{\mu\nu} = 2\epsilon g_{\mu\nu}, \quad (4.24)$$

which gives the infinitesimal *Weyl rescaling of the metric* tensor, see (3.28). Entry (2, 2) reads in components,

$$s_W \Gamma^\rho_{\mu\nu} = \delta^\rho_\nu \partial_\mu \epsilon + \delta^\rho_\mu \partial_\nu \epsilon + g^{\rho\lambda} \partial_\lambda \epsilon g_{\mu\nu}, \quad (4.25)$$



which is the infinitesimal *Weyl transformation of the Christoffel symbols*, of the Levi-Civita connection. Entry (1, 2) is,

$$s_W P_{\mu\nu} = \partial_\mu(\partial_\nu \epsilon) - \partial_\lambda \epsilon \Gamma_{\mu\nu}^\lambda = \nabla_\mu(\partial_\nu \epsilon), \quad (4.26)$$

which is the infinitesimal *Weyl transformation of the Schouten tensor*. Finally entry (2, 3) is,

$$s_W(g^{\rho\lambda} P_{\lambda\mu}) = -2\epsilon g^{\rho\lambda} P_{\lambda\mu} + g^{\rho\lambda} \left( \partial_\mu(\partial_\lambda \epsilon) - \Gamma_{\mu\lambda}^\alpha \partial_\alpha \epsilon \right). \quad (4.27)$$

This is redundant with (4.24) and (4.26).

Comparing with the finite transformations given in section 3.3 we see that the residual BRST algebra gives very easily the complete infinitesimal counterpart. Except for the Schouten tensor because the latter has a finite transformation which contains terms of order two in the Weyl parameter  $\epsilon$ . These terms are of course out of reach for the linear approximation captured by the BRST machinery.

The infinitesimal Weyl gauge transformation of the dressed normal curvature is given by  $s_W \Omega_0 = \Omega_0 \hat{v}_W - \hat{v}_W \Omega_0$ . Remembering this time that  $\epsilon$  commutes with even forms and using  $Wg^{-1} = -g^{-1}W^T$  (due to the  $\eta$ -skew symmetry of  $F_1$ ), we obtain

$$s_W \Omega_0 = \begin{pmatrix} 0 & s_W C & 0 \\ 0 & s_W W & s_W(g^{-1} \cdot C^T) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\partial \epsilon \cdot W & 0 \\ 0 & 0 & -g^{-1} \cdot W^T \cdot \partial \epsilon^T - 2\epsilon g^{-1} \cdot C^T \\ 0 & 0 & 0 \end{pmatrix}.$$

Using again the fact that  $s_W dx^\mu = 0$  we can write the entries in components. Entry (1, 2) gives,

$$s_W C_{\nu,\mu\sigma} = -\partial_\lambda \epsilon W_{\nu,\mu\sigma}^\lambda, \quad (4.28)$$

which is the infinitesimal *transformation of the Cotton tensor* under Weyl rescaling. Entry (2, 2) gives

$$s_W W_{\nu,\mu\sigma}^\rho = 0, \quad (4.29)$$

which states the *invariance of the Weyl tensor* under Weyl rescaling. Finally, entry (2, 3) is

$$s_W(g^{\rho\lambda} C_{\lambda,\mu\sigma}) = -2\epsilon g^{\rho\lambda} C_{\lambda,\mu\sigma} - g^{\rho\lambda} W_{\lambda,\mu\sigma}^\alpha \partial_\alpha \epsilon. \quad (4.30)$$

This is redundant with (4.28) and (4.24). Once more, it should be pointed out how easily the modified BRST algebra provides the complete infinitesimal counterparts of the finite transformations derived in section 3.3.

At last, the identity satisfied by the final composite ghost is,

$$s_W \hat{v} = -\hat{v}_W^2 \quad \rightarrow \quad \begin{pmatrix} s_W \epsilon & s_W(\partial \epsilon) & 0 \\ 0 & s_W \epsilon \delta & s_W(g^{-1} \partial \epsilon) \\ 0 & 0 & -s_W \epsilon \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\epsilon g^{-1} \partial \epsilon \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.31)$$

by recalling that  $\epsilon$  anticommutes with itself (the same holds for  $\partial\epsilon$ ). This just gives back the Weyl rescaling of the (inverse) metric  $s_W g^{-1} = -2\epsilon g^{-1}$  which is redundant with (4.24), but also

$$s_W \epsilon = 0, \quad (4.32)$$

which expresses the fact that the residual Weyl gauge group is abelian.

As a byproduct of this section, let us consider the *composite algebraic connection* introduced in Corollary 2. It reads,

$$\varpi_0 + \hat{v}_W = \begin{pmatrix} \epsilon & P + \partial\epsilon & 0 \\ dx & \Gamma + \epsilon \delta & g^{-1} \cdot (P + \partial\epsilon)^T \\ 0 & dx^T \cdot g & -\epsilon \end{pmatrix}. \quad (4.33)$$

Structurally, the algebraic connection is expressed as combinations between the metric  $g$  and the Weyl ghost  $\epsilon$  together with their derivatives. In [44–46] these combinations have been obtained through a completely different approach and turn out to be relevant for the algebraic study of the Weyl anomaly. Noteworthy, let us recall that in the present paper, we were able to use the well-tested BRST setting [29, 47] on the normal conformal Cartan connection, and we have subsequently modified the differential algebra by the dressing field method. The route followed is rather robust and gives a clear well-grounded geometrical picture from which these combinations naturally emerge.

## 5 Conclusion

In this paper, with a clear geometric view, we exhibited an example that extends the dressing field method given in [19] to the second order conformal structure for which two dressing operations have been performed. Their composition was secured by the fact that the two dressing fields fulfilled compatibility conditions regarding their transformation laws with respect to various symmetry sectors. The scheme can be generalized to any number of dressing fields, for instance to higher order  $G$ -structures [20].

In this example, treated in the very useful matrix notation, the final composite fields give the Riemannian parametrization of the normal conformal Cartan connection together with its curvature. The remaining symmetry has been shown to be the Weyl rescalings. The residual transformation of the composite fields provides at once well-known conformal transformations of noticeable tensors in conformal geometry of  $\mathcal{M}$ , namely, the (pseudo-Riemannian) metric, the Schouten, Cotton and Weyl tensors and the Christoffel symbols of the Levi-Civita connection. This is summarized in table 1. In short, the dressing scheme provides gauge like transformation recombinations of the fields which amount to eliminating spurious degrees of freedom to the benefit of geometrical objects together with their properties under the Weyl symmetry.

Let us notice that expressions (3.11) of the normal conformal Cartan connection and its curvature, after the dressing by  $u_1$ , turn out to be the “tractor connection” and its curvature

	Starting geometry		Outcoming geometry	
	Second order conformal structure	Degrees of freedom	Natural geometry	Degrees of freedom
(1) Variables	$\varpi \in \Omega^1(\mathbf{U}, \mathfrak{g})$	$m\left(1 + \frac{m(m-1)}{2} + 2m\right)$	$g_{\mu\nu}$ $\Gamma^\rho_{\mu\nu}$ $P_{\mu\nu}$	$\frac{m(m+1)}{2}$ $m^3$ 0
(2) Symmetries	$(\text{SO} \times \mathbf{W}) \mathbb{R}^{m*}$	$\frac{m(m-1)}{2} + 1 + m$	$\mathbf{W} = \mathbb{R}_+ \setminus \{0\}$	1
(3) Constraints	$\Theta = 0$	$m \frac{m(m-1)}{2}$	$\nabla g = 0$	$m \frac{m(m+1)}{2}$
	$\text{Ric}(\mathbf{F})=0$	$\frac{m(m+1)}{2}$	$T = 0$	$m \frac{m(m-1)}{2}$
	$f = 0$	$\frac{m(m-1)}{2}$		
Total degrees of freedom (1)–(2)–(3)		$\frac{m(m+1)}{2} - 1$		$\frac{m(m+1)}{2} - 1$

**Table 1.** Counting the degrees of freedom of the *normal* conformal geometry before and after the dressing operation. Remark that the normal geometry is the most natural one since its total degrees of freedom are those of the conformal class  $[g]$  of the metric  $g$ .

given in [48]. This suggests that the dressing field method ought to be implemented in the language of tractor calculus related to parabolic Cartan geometries.<sup>3</sup>

The BRST counterpart of the dressing field method has been exhibited. Its central object is the *composite ghost* which corresponds to a “field dependent change of generators” and encodes the residual gauge symmetry of the composite field, if any. Applied to the second order conformal structure, the composite algebraic connection (composite field + composite ghost) is shown to give a structural geometric interpretation of the cohomological results obtained in [44–46] regarding the Weyl symmetry. Moreover the corresponding *modified BRST algebra* provides in a very effective manner the linearized version of the residual gauge symmetry of Weyl rescalings derived in the first part of the paper.

As already noticed in [19], the dressing field method accomodates different notion of connections (Ehresmann, Cartan or some generalized connections): this makes comparisons easier between various examples of interest. For instance, a parallel can be made between the electroweak part of the standard model [19, 24] and the conformal Cartan structure. The erased subgroup  $\text{SU}(2)$  is the mirror of the Lorentz and special conformal transformations, while the abelian residual gauge symmetry  $\text{U}(1)$  falls together with the Weyl dilations.

All the computations were performed at the classical level and were governed by geometrical considerations. It deserves to see how all the process goes through the quantiza-

<sup>3</sup>It is expected that applying the dressing field method in this context would give rise to the same results, but, since the tractor calculus uses the language of differential operators, they may appear in a more cumbersome formulation.

tion, in particular the question of the quantum version of the composite fields and how to manage the erasure of gauge sub-symmetries.

Furthermore, one could push further ahead the use of the BRST techniques by combining together conformal gauge transformations and diffeomorphisms (of  $\mathcal{M}$ ). This raises the question of the compatibility with the dressing field method in order to reduce the whole mixed symmetry to the  $\text{Diff} \times \text{Weyl}$  symmetry with the goal to recover some of the results obtained in [44, 45]. This issue will be addressed elsewhere [49] in a companion paper.

## Acknowledgments

We would like to thank R. Stora and T. Schücker for fruitful discussions at the earlier stages of this work. We are grateful to one of the referees for suggestions improving the paper and for pointing out to us references on tractor calculus.

This work has been carried out in the framework of the Labex ARCHIMEDE (Grant No. ANR-11-LABX-0033) and of the A\*MIDEX project (Grant No. ANR-11-IDEX-0001-02), funded by the “Investissements d’Avenir” French Government program managed by the French National Research Agency (ANR).

## A Weyl transformations of the first composite fields

For the sake of completeness, this appendix concerns the Weyl transformations (both finite and BRST versions) on the first composite fields  $\varpi_1$  and  $\Omega_1$  obtained in (3.5) and (3.6) respectively. Their finite transformations are computed similarly to (3.22) upon using  $u_1^W$  given in (3.26). A straightforward matrix computation yields, on the one hand,

$$\begin{aligned}\varpi_1^W &:= (\varpi^{u_1})^W = (\varpi^W)^{u_1^W} = (\varpi_1^W)^{W^{-1}u_1k_1W} \\ &= \varpi_1^{k_1W} = (k_1W)^{-1}\varpi_1(k_1W) + (k_1W)^{-1}d(k_1W)\end{aligned}\quad (\text{A.1})$$

$$\begin{pmatrix} 0 & \alpha_1^W & 0 \\ \theta^W & A_1^W & \alpha_1^{tW} \\ 0 & \theta^{tW} & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^{-1}(\alpha_1 + D(\zeta \cdot e^{-1}) - (\zeta \cdot e^{-1})\theta(\zeta \cdot e^{-1}) + \frac{1}{2}(\zeta \cdot e^{-1})(\zeta \cdot e^{-1})^t\theta^t) & 0 \\ z\theta & A_1 + \theta(\zeta \cdot e^{-1}) - (\zeta \cdot e^{-1})^t\theta^t & * \\ 0 & z\theta^t & 0 \end{pmatrix}$$

where  $D(\zeta \cdot e^{-1}) = d(\zeta \cdot e^{-1}) - (\zeta \cdot e^{-1})A_1$  is the covariant derivative with respect to the spin connection  $A_1$ , where we have set  $\zeta = z^{-1}\partial z$  as in the main text, and where  $*$  = (entry(1,2))<sup>t</sup>.

On the other hand,

$$\begin{aligned}\Omega_1^W &:= \begin{pmatrix} f_1^W & \Pi_1^W & 0 \\ \Theta_1^W & F_1^W & \Pi_1^{tW} \\ 0 & \Theta_1^{tW} & -f_1^W \end{pmatrix} = (\Omega^W)^{u_1^W} = (\Omega_1^W)^{W^{-1}u_1k_1W} = \Omega_1^{k_1W} = (k_1W)^{-1}\Omega_1k_1W \quad (\text{A.2}) \\ &= \begin{pmatrix} f_1 - (\zeta \cdot e^{-1})\Theta_1 & z^{-1}(\Pi_1 - (\zeta \cdot e^{-1})(F_1 - f_1) - (\zeta \cdot e^{-1})\Theta_1(\zeta \cdot e^{-1}) + \frac{1}{2}(\zeta \cdot e^{-1})(\zeta \cdot e^{-1})^t\Theta_1^t) & 0 \\ z\Theta_1 & F_1 + \Theta_1(\zeta \cdot e^{-1}) - (\zeta \cdot e^{-1})^t\Theta_1^t & * \\ 0 & z\Theta_1^t & * \end{pmatrix}\end{aligned}$$

where entry (2,3) = (entry (1,2))<sup>t</sup> and entry (3,3) = − entry (1,1).

In the *normal* case, see (3.10), formula (A.1) remains formally unchanged but  $\alpha_1$  becomes the Schouten 1-form, and the Weyl variation (A.2) reduces to (see (3.6))

$$\Omega_1^W = \begin{pmatrix} 0 & \Pi_1^W & 0 \\ 0 & F_1^W & \Pi_1^{tW} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^{-1}(\Pi_1 - (\zeta \cdot e^{-1})F_1) & 0 \\ 0 & F_1 & (\text{entry}(1,2))^t \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.3})$$

with  $\Pi_1 = d\alpha_1 + \alpha_1 A_1$  is the Cotton 2-form, and  $F_1 = dA_1 + A_1^2 + \theta\alpha_1 - (\theta\alpha_1)^t$  is the Weyl 2-form.

Let us now turn to the corresponding BRST setting. It comes from the modification of the initial BRST algebra by the dressing field  $u_1$ . The matrix avatar of the Weyl ghost  $\epsilon$  is given by setting  $v_L \equiv 0$  in the ghost (4.16) and we keep the same notation  $\hat{v}_1$ . For the composite field  $\varpi_1$ , one has thus

$$s_W \varpi_1 = -d\hat{v}_1 - \hat{v}_1 \varpi_1 - \varpi_1 \hat{v}_1$$

$$\begin{pmatrix} 0 & s_W \alpha_1 & 0 \\ s_W \theta & s_W A_1 & s_W \alpha_1^t \\ 0 & s_W \theta^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d(\partial\epsilon \cdot e^{-1}) - (\partial\epsilon \cdot e^{-1})A_1 - \epsilon\alpha_1 & 0 \\ \epsilon\theta & -\theta(\partial\epsilon \cdot e^{-1}) - (\partial\epsilon \cdot e^{-1})^t \theta^t & * \\ 0 & \epsilon\theta^t & 0 \end{pmatrix} \quad (\text{A.4})$$

where  $*$  = (entry (1,2))<sup>t</sup>. For its curvature  $\Omega_1$

$$s_W \Omega_1 = \Omega_1 \hat{v}_1 - \hat{v}_1 \Omega_1$$

$$\begin{pmatrix} 0 & s_W \Pi_1 & 0 \\ 0 & s_W F_1 & s_W \Pi_1^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon\Pi_1 - (\partial\epsilon \cdot e^{-1})F_1 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{A.5})$$

where  $*$  = (entry (1,2))<sup>t</sup>.

These are the Weyl transformations of the normal conformal Cartan connection in the internal Minkowski indices before the second dressing by  $u_0$ . As seen in the main text, the dressing field  $u_0$  allows to switch to spacetime (greek) indices.

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